

CHAINS OF THEORIES AND COMPANIONABILITY

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ABSTRACT. The theory of fields that are equipped with a countably infinite family of commuting derivations is not companionable; but if the axiom is added whereby the characteristic of the fields is zero, then the resulting theory is companionable. Each of these two theories is the union of a chain of companionable theories. In the case of characteristic zero, the model-companions of the theories in the chain form another chain, whose union is therefore the model-companion of the union of the original chain. However, in a signature with predicates, in all finite numbers of arguments, for linear dependence of vectors, the two-sorted theory of vector-spaces with their scalar-fields is companionable, and it is the union of a chain of companionable theories, but the model-companions of the theories in the chain are mutually inconsistent. Finally, the union of a chain of non-companionable theories may be companionable.

A **theory** in a given signature is a set of sentences, in the first-order logic of that signature, that is closed under logical implication. We shall consider chains $(T_m : m \in \omega)$ of theories: this means

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \quad (*)$$

The signature of T_m will be \mathcal{S}_m , so automatically $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots$.

In one motivating example, \mathcal{S}_m is $\{0, 1, -, +, \cdot, \partial_0, \dots, \partial_{m-1}\}$, the signature of fields with m additional singular operation-symbols; and T_m is m -DF, the theory of fields (of any characteristic) with m commuting derivations. In this example, each T_{m+1} is a **conservative extension** of T_m , that is, $T_{m+1} \supseteq T_m$ and every sentence in T_{m+1} of signature \mathcal{S}_m is already in T_m . We establish this by showing that every model of T_m expands to a model of T_{m+1} . (This condition is sufficient, but not necessary [3, §2.6, exer. 8, p. 66].) If $(K, \partial_0, \dots, \partial_{m-1}) \models m$ -DF, then $(K, \partial_0, \dots, \partial_m) \models (m+1)$ -DF, where ∂_m is the 0-derivation.

The union of the theories m -DF can be denoted by ω -DF: it is the theory of fields with ω -many commuting derivations. Each of the theories m -DF has a *model-companion*, called m -DCF [11]; but we shall show (as Theorem 3 below) that ω -DF has no model-companion. Let us recall that a **model-companion** of a theory T is a theory T^* in the same signature such that (1) $T_\forall = T^*_\forall$, that is, every model of one of the theories embeds in a model of the other, and (2) T^* is **model-complete**, that is, $T^* \cup \text{diag}(\mathfrak{M})$ axiomatizes a complete theory for all models \mathfrak{M} of T^* . Here $\text{diag}(\mathfrak{M})$ is the quantifier-free theory of \mathfrak{M} with parameters: equivalently, $\text{diag}(\mathfrak{M})$ is the theory of all structures in which \mathfrak{M}_M embeds. (These notions, with historical references, are reviewed further in [11].) A theory has at most one model-companion, by an argument with interwoven elementary chains.

Let m -DF₀ be m -DF with the additional requirement that the field have characteristic 0. Then m -DF₀ has a model-companion, called m -DCF₀ [6]. We shall show (as Theorem 6 below) that m -DCF₀ \subseteq $(m+1)$ -DCF₀. It will follow then that the union ω -DF₀ of the m -DF₀ has a model-companion, which is the union of the

Date: March 28, 2013.

2010 *Mathematics Subject Classification.* 03C10, 03C60, 12H05, 13N15.

$m\text{-DCF}_0$. This is by the following general result, which has been observed also by Alice Medvedev [7, 8]. Again, the theories T_k are as in (*) above.

Theorem 1. *Suppose each theory T_k has a model-companion T_k^* , and*

$$T_0^* \subseteq T_1^* \subseteq T_2^* \subseteq \cdots \quad (\dagger)$$

Then the theory $\bigcup_{k \in \omega} T_k$ has a model-companion, namely $\bigcup_{k \in \omega} T_k^$.*

Proof. Write U for $\bigcup_{k \in \omega} T_k$, and U^* for $\bigcup_{k \in \omega} T_k^*$. Suppose $\mathfrak{A} \models U$, and Γ is a finite subset of $U^* \cup \text{diag}(\mathfrak{A})$. Then Γ is a subset of $T_k^* \cup \text{diag}(\mathfrak{A} \upharpoonright \mathcal{S}_k)$ for some k in ω , and also $\mathfrak{A} \upharpoonright \mathcal{S}_k \models T_k$. Since $(T_k^*)_{\forall} \subseteq T_k$, the structure $\mathfrak{A} \upharpoonright \mathcal{S}_k$ must embed in a model of T_k^* ; and this model will be a model of Γ . We conclude that Γ is consistent. Therefore $U^* \cup \text{diag}(\mathfrak{A})$ is consistent. Thus $U^*_{\forall} \subseteq U$. By symmetry $U_{\forall} \subseteq U^*$.

Similarly, if $\mathfrak{B} \models U^*$, then $T_k^* \cup \text{diag}(\mathfrak{B} \upharpoonright \mathcal{S}_k)$ axiomatizes a complete theory in each case, and therefore $U^* \cup \text{diag}(\mathfrak{B})$ is complete. \square

The foregoing proof does not require that the signatures \mathcal{S}_k form a chain, but needs only that every finite subset of $\bigcup_{k \in \omega} \mathcal{S}_k$ be included in some \mathcal{S}_k . This is the setting for Medvedev's [8, Prop. 2.4, p. 6], which then has the same proof as the foregoing. Also in Medvedev's setting, each T_{k+1}^* is a conservative extension of T_k^* ; but only the weaker assumption $T_k^* \subseteq T_{k+1}^*$ is needed in the proof.

Medvedev notes that many properties that the theories T_k might have are 'local' and are therefore preserved in $\bigcup_{k \in \omega} T_k$: examples are completeness, elimination of quantifiers, stability, and simplicity. In her main application, \mathcal{S}_n is the signature of fields with singular operation-symbols $\sigma_{m/n!}$, where $m \in \mathbb{Z}$; and T_n is the theory of fields on which the $\sigma_{m/n!}$ are automorphisms such that

$$\sigma_{k/n!} \circ \sigma_{m/n!} = \sigma_{(k+m)/n!}.$$

Then T_n includes the theory S_n of fields with the single automorphism $\sigma_{1/n!}$. Using [12, §1] (which is based on [3, ch. 5]), we may observe at this point that reduction of models of T_n to models of S_n is actually an equivalence of the categories $\text{Mod}^{\subseteq}(T_n)$ and $\text{Mod}^{\subseteq}(S_n)$, whose objects are models of the indicated theories, and whose morphisms are embeddings. We thus have at hand a (rather simple) instance of the hypothesis of the following theorem.

Theorem 2. *Suppose (I, J) is a bi-interpretation of theories S and T such that I is an equivalence of the categories $\text{Mod}^{\subseteq}(S)$ and $\text{Mod}^{\subseteq}(T)$. If S has the model-companion S^* , and $S \subseteq S^*$, then T also has a model-companion, which is the theory of those models \mathfrak{B} of T such that $J(\mathfrak{B}) \models S^*$.*

Proof. The class of models \mathfrak{B} of T such that $J(\mathfrak{B}) \models S^*$ is elementary. Let T^* be its theory. Then $T \subseteq T^*$. Suppose $\mathfrak{B} \models T$. Then $J(\mathfrak{B}) \models S$, so $J(\mathfrak{B})$ embeds in a model \mathfrak{A} of S^* . Consequently $I(J(\mathfrak{B}))$ embeds in $I(\mathfrak{A})$. Also $I(\mathfrak{A}) \models T^*$, since $\mathfrak{A} \cong J(I(\mathfrak{A}))$. Since also $\mathfrak{B} \cong I(J(\mathfrak{B}))$, we conclude that \mathfrak{B} embeds in a model of T^* . Finally, T^* is model-complete. Indeed, suppose now \mathfrak{B} and \mathfrak{C} are models of T^* such that $\mathfrak{B} \subseteq \mathfrak{C}$. An embedding of $J(\mathfrak{B})$ in $J(\mathfrak{C})$ is induced, and these structures are models of S^* , so the embedding is elementary. Therefore the induced embedding of $I(J(\mathfrak{B}))$ in $I(J(\mathfrak{C}))$ is also elementary. By the equivalence of the categories, $\mathfrak{B} \preceq \mathfrak{C}$. \square

In the present situation, the theory S_n has a model-companion [5, 1]; let us denote this by ACFA_n . By the theorem then, T_n has a model-companion T_n^* , which is axiomatized by $T_n \cup \text{ACFA}_n$. We have $\text{ACFA}_n \subseteq T_{n+1}^*$ by [1, 1.12, Cor. 1, p. 3013]. By Theorem 1 then, $\bigcup_{n \in \omega} T_n$ has a model-companion, which is the union of the T_n^* . Medvedev calls this union $\mathbb{Q}\text{ACFA}$; she shows for example that it

preserves the simplicity of the ACFA_n , as noted above, though it does not preserve their supersimplicity.

The following is similar to the result that the theory of fields with a derivation *and* an automorphism (of the field-structure only) has no model-companion [10]. The obstruction lies in positive characteristics p , where all derivatives of elements with p -th roots must be 0.

Theorem 3. *The theory ω -DF has no model-companion.*

Proof. We use that an $\forall\exists$ theory T has a model-companion if and only if the class of its *existentially closed* models is elementary, and in this case the model-companion is the theory of this class [2]. (A model \mathfrak{A} of T is an **existentially closed** model, provided that if $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preceq_1 \mathfrak{B}$, that is, all quantifier-free formulas over A that are soluble in \mathfrak{B} are soluble in \mathfrak{A} .) For each n in ω , the theory ω -DF has an existentially closed model \mathfrak{A}_n , whose underlying field includes $\mathbb{F}_p(\alpha)$, where α is transcendental; and in this model,

$$\partial_k \alpha = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then α has no p -th root in \mathfrak{A}_n . Therefore, in a non-principal ultraproduct of the \mathfrak{A}_n , α has no p -th root, although $\partial_n \alpha = 0$ for all n in ω , so that α does have a p -th root in some extension. Thus the ultraproduct is not an existentially closed model of ω -DF. Therefore the class of existentially closed models of ω -DF is not elementary. \square

It follows then by Theorem 1 that m -DCF $\not\leq (m+1)$ -DCF for at least one m . In fact this is so for all m , since

$$m\text{-DCF} \vdash p = 0 \rightarrow \forall x \left(\bigwedge_{i < m} \partial_i x = 0 \rightarrow \exists y y^p = x \right),$$

but $(m+1)$ -DCF does not entail this sentence, since

$$(m+1)\text{-DCF} \vdash \exists x \left(\bigwedge_{i < m} \partial_i x = 0 \wedge \partial_m x \neq 0 \right).$$

However, this observation by itself is not enough to establish the last theorem. For, by the results of [12], it is possible for each T_k to have a model-companion T_k^* , while $\bigcup_{k \in \omega} T_k$ has a model-companion that is not $\bigcup_{k \in \omega} T_k^*$. We may even require T_{k+1} to be a conservative extension of T_k .

Indeed, if $k > 0$, then in the notation of [12], VS_k is the theory of vector-spaces with their scalar-fields in the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *, P^k\}$, where \circ is multiplication of scalars, and $*$ is the action of the scalar-field on the vector-space, and P^k is k -ary linear dependence. In particular, P^2 may be written also as $\|$. Then VS_k has a model-companion, VS_k^* , which is the theory of k -dimensional vector-spaces over algebraically closed fields [12, Thm 2.3]. Let $\text{VS}_\omega = \bigcup_{1 \leq k < \omega} \text{VS}_k$. (This was called VS_∞ in [12].) This theory has the model-companion VS_ω^* , which is the theory of infinite-dimensional vector-spaces over algebraically closed fields [12, Thm 2.4]. In particular VS_ω^* is not the union of the VS_k^* , because these are mutually inconsistent. We now turn this into a result about chains:

Theorem 4. *If $1 \leq n < \omega$, let T_n be the theory axiomatized by $\text{VS}_1 \cup \dots \cup \text{VS}_n$. Then T_n has a model-companion T_n^* , which is axiomatized by $T_n \cup \text{VS}_n^*$. Also T_{n+1} is a conservative extension of T_n . However, the model-companion VS_ω^* of the union VS_ω of the chain $(T_n : 1 \leq n < \omega)$ is not the union of the T_n^* .*

Proof. Every vector-space can be considered as a model of every VS_k and hence of every T_k . In particular, T_{n+1} is a conservative extension of T_n . If the theories T_n^* are as claimed, then they are mutually inconsistent, and so VS_ω^* is not their union. It remains to show that there are theories T_n^* as claimed. We already know this when $n = 1$. For the other cases, if $1 \leq k < n$, we define the relations P^k in models of VS_n of dimension at least n .

Let VS_n^m the theory of such models: that is, VS_n^m is axiomatized by VS_n and the requirement that the space have dimension at least n . The relation P^1 is defined in models of VS_n^m (and indeed in models of VS_n) by the quantifier-free formula $\mathbf{x} = \mathbf{0}$. If $n > 2$, then there are existential formulas that, in each model of VS_n^m , define the relation \parallel and its complement [12, §2, p. 431]. More generally, if $1 \leq k < n - 1$, then, using existential formulas, we can define P^{k+1} and its complement in models of $T_k \cup \text{VS}_n^m$ or just $\text{VS}_k \cup \text{VS}_n^m$. Indeed, $\neg P^{k+1} \mathbf{x}_0 \cdots \mathbf{x}_k$ is equivalent to $\exists(\mathbf{x}_{k+1}, \dots, \mathbf{x}_{n-1}) \neg P^n \mathbf{x}_0 \cdots \mathbf{x}_{n-1}$, and $P^{k+1} \mathbf{x}_0 \cdots \mathbf{x}_k$ is equivalent to

$$\exists(\mathbf{x}_{k+1}, \dots, \mathbf{x}_n) \left(P^k \mathbf{x}_1 \cdots \mathbf{x}_k \vee \left(\neg P^n \mathbf{x}_1 \cdots \mathbf{x}_n \wedge \bigwedge_{j=k+1}^n P^n \mathbf{x}_0 \cdots \mathbf{x}_{j-1} \mathbf{x}_{j+1} \cdots \mathbf{x}_n \right) \right).$$

For, in a space of dimension at least n , if $(\mathbf{a}_0, \dots, \mathbf{a}_k)$ is linearly dependent, but $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ is not, this means precisely that $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is independent for some $(\mathbf{a}_{k+1}, \dots, \mathbf{a}_n)$, but \mathbf{a}_0 is a *unique* linear combination of $(\mathbf{a}_1, \dots, \mathbf{a}_n)$, and in fact of $(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_n)$ whenever $k+1 \leq j \leq n$, and (therefore) of $(\mathbf{a}_1, \dots, \mathbf{a}_k)$.

By [12, Lem 1.1, 1.2], if $1 \leq k < n - 1$, we now have that reduction from models of $T_{k+1} \cup \text{VS}_n^m$ to models of $T_k \cup \text{VS}_n^m$ is an equivalence of the categories $\text{Mod}^\subseteq(T_{k+1} \cup \text{VS}_n^m)$ and $\text{Mod}^\subseteq(T_k \cup \text{VS}_n^m)$. Combining these results for all k , we have that reduction from models of $T_{n-1} \cup \text{VS}_n^m$ to models of VS_n^m is an equivalence of the categories $\text{Mod}^\subseteq(T_{n-1} \cup \text{VS}_n^m)$ and $\text{Mod}^\subseteq(\text{VS}_n^m)$. Since $\text{VS}_n \subseteq \text{VS}_n^m$ and every model of VS_n embeds in a model of VS_n^m , the two theories have the same model-companion, namely VS_n^* . Similarly, T_n and $T_{n-1} \cup \text{VS}_n^m$ have the same model-companion; and by Theorem 2, this is axiomatized by $T_n \cup \text{VS}_n^*$. \square

A one-sorted version of the last theorem can be developed as follows. Let VS_n^r comprise the sentences of VS_n^m having one-sorted signature $\{\mathbf{0}, -, +, P^n\}$ of the sort of vectors alone. It is not obvious that all models of VS_n^m can be furnished with scalar-fields to make them models of VS_n^r again; but this will be the case. By [12, Thm 1.1], it is the case when $n = 2$: reduction of models of VS_2^m to models of VS_2^r is an equivalence of the categories $\text{Mod}^\subseteq(\text{VS}_2^m)$ and $\text{Mod}^\subseteq(\text{VS}_2^r)$. This reduction is therefore **conservative**, by the definition of [12, p. 426]. It is said further at [12, p. 431] that reduction from VS_n^m to VS_n^r is conservative when $n > 2$; but the details are not spelled out. However, the claim can be established as follows. Immediately, reduction from $\text{VS}_2 \cup \text{VS}_n^m$ to $\text{VS}_2^r \cup \text{VS}_n^r$ is conservative. In particular, models of the latter set of sentences really are vector-spaces without their scalar-fields. It is noted in effect in the proof of Theorem 4 that reduction from $\text{VS}_2 \cup \text{VS}_n^m$ to VS_n^m is conservative. Furthermore, in models of the latter theory, the defining of parallelism and its complement is done with existential formulas *in the signature of vectors alone*. Therefore reduction from $\text{VS}_2^r \cup \text{VS}_n^r$ to VS_n^r is conservative. We now have the following commutative diagram of reduction-functors, three of them

being conservative, that is, being equivalences of categories.

$$\begin{array}{ccc} \text{Mod}^\subseteq(\text{VS}_2 \cup \text{VS}_n^m) & \longrightarrow & \text{Mod}^\subseteq(\text{VS}_n^m) \\ \downarrow & & \downarrow \\ \text{Mod}^\subseteq(\text{VS}_2^r \cup \text{VS}_n^r) & \longrightarrow & \text{Mod}^\subseteq(\text{VS}_n^r) \end{array}$$

Therefore the remaining reduction, from VS_n^m to VS_n^r , must be conservative.

Now there is a version of Theorem 4 where T_n is axiomatized by $\text{VS}_2^r \cup \dots \cup \text{VS}_n^r$. Indeed, by Theorem 2, T_n has a model-companion, which is the theory (in the same signature) of n -dimensional vector-spaces over algebraically closed fields; and the union of the T_n has a model-companion, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields; but this theory is not the union of the model-companions of the T_n .

The implication $A \Rightarrow B$ in the following is used implicitly at [1, 1.12, p. 3013] to establish the result used above, that if (K, σ) is a model of ACFA, then so is (K, σ^m) , assuming $m \geq 1$.

Theorem 5. *Assuming as usual $T_0 \subseteq T_1$, where each T_k has signature \mathcal{S}_k , we consider the following conditions.*

A. *For every model \mathfrak{A} of T_1 and model \mathfrak{B} of T_0 such that*

$$\mathfrak{A} \upharpoonright \mathcal{S}_0 \subseteq \mathfrak{B}, \quad (\ddagger)$$

there is a model \mathfrak{C} of T_1 such that

$$\mathfrak{A} \subseteq \mathfrak{C}, \quad \mathfrak{B} \subseteq \mathfrak{C} \upharpoonright \mathcal{S}_0. \quad (\S)$$

B. *The reduct to \mathcal{S}_0 of every existentially closed model of T_1 is an existentially closed model of T_0 .*

C. *T_0 has the Amalgamation Property: if one model embeds in two others, then those two in turn embed in a fourth model, compatibly with the original embeddings.*

D. *T_1 is $\forall\exists$ (so that every model embeds in an existentially closed model).*

We have the two implications

$$A \implies B, \quad B \ \& \ C \ \& \ D \implies A,$$

but there is no implication among the four conditions that does not follow from these. This is true, even if T_1 is required to be a conservative extension of T_0 .

Proof. Suppose A holds. Let \mathfrak{A} be an existentially closed model of T_1 , and let \mathfrak{B} be an arbitrary model of T_0 such that (\ddagger) holds. By hypothesis, there is a model \mathfrak{C} of T_1 such that (\S) holds. Then $\mathfrak{A} \preceq_1 \mathfrak{C}$, and therefore $\mathfrak{A} \upharpoonright \mathcal{S}_0 \preceq_1 \mathfrak{C} \upharpoonright \mathcal{S}_0$, and *a fortiori* $\mathfrak{A} \upharpoonright \mathcal{S}_0 \preceq_1 \mathfrak{B}$. Therefore $\mathfrak{A} \upharpoonright \mathcal{S}_0$ must be an existentially closed model of T_0 . Thus B holds.

Suppose conversely B holds, along with C and D. Let $\mathfrak{A} \models T_1$ and $\mathfrak{B} \models T_0$ such that (\ddagger) holds. We establish the consistency of $T_1 \cup \text{diag}(\mathfrak{A}) \cup \text{diag}(\mathfrak{B})$. It is enough to show the consistency of

$$T_1 \cup \text{diag}(\mathfrak{A}) \cup \{\exists x \varphi(x)\}, \quad (\P)$$

where φ is an arbitrary quantifier-free formula of $\mathcal{S}_0(A)$ that is soluble in \mathfrak{B} . By D, there is an existentially closed model \mathfrak{C} of T_1 that extends \mathfrak{A} . By B then, $\mathfrak{C} \upharpoonright \mathcal{S}_0$ is an existentially closed model of T_0 that extends $\mathfrak{A} \upharpoonright \mathcal{S}_0$. By C, both \mathfrak{B} and $\mathfrak{C} \upharpoonright \mathcal{S}_0$ embed over $\mathfrak{A} \upharpoonright \mathcal{S}_0$ in a model of T_0 . In particular, φ will be soluble in this model. Therefore φ is already soluble in $\mathfrak{C} \upharpoonright \mathcal{S}_0$ itself. Thus \mathfrak{C} is a model of (\P) . Therefore A holds.

The foregoing arguments eliminate the five possibilities marked X on the table below, where 0 means false, and 1, true. We give examples of each of the remaining

	1	X	2	3	4	X	5	6	7	X	8	9	10	X	X	11
A	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
C	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
D	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

cases, numbered according to the table. In each example, T_0 will be the reduct of T_1 to \mathcal{S}_0 . We shall denote by \mathcal{S}_f the signature $\{+, \cdot, -, 0, 1\}$ of fields; and by \mathcal{S}_{vs} , the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *\}$ of vector-spaces as two-sorted structures.

1. We first give an example in which none of the four lettered conditions hold. Let $\mathcal{S}_0 = \mathcal{S}_f \cup \{a, b\}$ and $\mathcal{S}_1 = \mathcal{S}_0 \cup \{c\}$. Let T_1 be the theory of fields of characteristic p with distinguished elements a, b , and c such that $\{a, c\}$ or $\{b, c\}$ is p -independent, and if $\{b, c\}$ is p -independent, then so is $\{b, c, d\}$ for some d . Then T_0 is the theory of fields of characteristic p in which, for some c , $\{a, c\}$ or $\{b, c\}$ is p -independent, and if $\{b, c\}$ is p -independent, then so is $\{b, c, d\}$ for some d . The negations of the four lettered conditions are established as follows. Throughout, a, b, c , and d will be algebraically independent over \mathbb{F}_p .

$\neg A$. We have

$$(\mathbb{F}_p(a, b^{1/p}, c), a, b, c) \models T_1, \quad (\mathbb{F}_p(a, b^{1/p}, c^{1/p}), a, b) \models T_0,$$

but if $(\mathbb{F}_p(a, b^{1/p}, c), a, b, c)$ is a substructure of a model (K, a, b, c) of T_1 , then K cannot contain $c^{1/p}$.

$\neg B$. T_0 has no existentially closed models, since an element of a model that is p -independent from a or b will always have a p -th root in some extension. Similarly, no model of T_1 in which $\{a, c\}$ is not p -independent is existentially closed. But T_1 does have existentially closed models, which are just the separably closed fields of characteristic p with p -basis $\{a, c\}$ and with an additional element b .

$\neg C$. T_0 does not have the Amalgamation Property, since $(\mathbb{F}_p(a, b^{1/p}, c), a, b)$ and $(\mathbb{F}_p(a^{1/p}, b, c, d), a, b)$ are models that do not embed in the same model over the common substructure $(\mathbb{F}_p(a, b, c), a, b)$, which is a model of T_0 .

$\neg D$. T_1 is not $\forall\exists$, since, as we have already noted, models in which $\{a, c\}$ is not p -independent do not embed in existentially closed models.

2. For an example of the column headed by 2 in the table, we let \mathcal{S}_0 and \mathcal{S}_1 be as in 1; but now T_1 is the theory of fields of characteristic p with distinguished elements a, b , and c such that $\{a, c, d\}$ or $\{b, c, d\}$ is p -independent for some d . This ensures that T_1 has no existentially closed models, so B holds vacuously; but the other three conditions still fail.

3. T_0 and T_1 are the same theory, so A and B hold trivially; and this theory is the theory of vector-spaces of dimension at least 2, in the signature \mathcal{S}_{vs} , so the theory neither has the Amalgamation Property, nor is $\forall\exists$.

4. T_1 is DF_p with the additional requirement that the field have p -dimension at least 2; and $\mathcal{S}_0 = \mathcal{S}_f$, so T_0 is the theory of fields of characteristic p with p -dimension at least 2. The latter theory has the Amalgamation Property; but the other conditions fail. Indeed, let $(\mathbb{F}_p(a, b), D)$ be the model of T_1 in which $Da = 1$ and $Db = 0$: then the field $\mathbb{F}_p(a, b)$ embeds in $\mathbb{F}_p(a^{1/p}, b)$, which is a model of T_0 , but D does not extend to this field. Also, T_0 has no existentially closed models; but T_1 does, and indeed it has a model-companion, namely DCF_p . Also T_1 is not $\forall\exists$, since T_0 is not: there is a chain of models of the latter, whose union is not a

model, and we can make the structures in the chain into models of T_1 by adding the zero derivation.

5. $\mathcal{S}_0 = \mathcal{S}_f$, and $\mathcal{S}_1 = \mathcal{S}_0 \cup \{a\}$. T_1 is the theory of fields of characteristic p with distinguished element a , which is p -independent from another element; so T_0 is (as in 4) the theory of fields of characteristic p with p -dimension at least 2. Then we already have that C holds. But A fails: just let \mathfrak{A} be $(\mathbb{F}_p(a, b), a)$, and let \mathfrak{B} be $\mathbb{F}_p(a^{1/p}, b)$. Also T_1 has no existentially closed models, so B holds trivially, but T_1 is not $\forall\exists$.

6. T_0 and T_1 are the same, namely the theory of fields of characteristic p of positive p -dimension, in the signature of fields, so this theory has the Amalgamation Property, but is not $\forall\exists$.

7. $\mathcal{S}_0 = \mathcal{S}_{vs}$, $\mathcal{S}_1 = \mathcal{S}_0 \cup \{\|\mathbf{a}, \mathbf{b}\}\}$, and T_1 is axiomatized by $VS_2 \cup \{\mathbf{a} \nparallel \mathbf{b}\}$, so it is $\forall\exists$. Then T_0 is the theory of vector-spaces of dimension at least 2. As in Theorem 4 above, T_1 has a model-companion, namely the theory of vector-spaces over algebraically closed fields with basis $\{\mathbf{a}, \mathbf{b}\}$. But T_0 has no existentially closed models, since for all independent vectors \mathbf{a} and \mathbf{b} in some model, the equation

$$x * \mathbf{a} + y * \mathbf{b} = \mathbf{0} \quad (||)$$

is always soluble in some extension. Thus B fails. Then T_0 also does not have the Amalgamation Property, since the solutions of $(||)$ may satisfy $2x^2 = y^2$ in one extension, but $3x^2 = y^2$ in another. Similarly, A fails, since the reduct to \mathcal{S}_0 of a model of T_1 may embed in a model of T_0 in which \mathbf{a} and \mathbf{b} are parallel.

8. $\mathcal{S}_0 = \mathcal{S}_{vs} \cup \{\|\mathbf{a}, \mathbf{b}\}\}$, $\mathcal{S}_1 = \mathcal{S}_0 \cup \{\mathbf{a}, \mathbf{b}\}$, and T_1 is axiomatized by VS_2 together with

$$\forall x \forall y (x * \mathbf{a} + y * \mathbf{b} = \mathbf{0} \rightarrow 2x^2 = y^2). \quad (**)$$

Then T_0 is the theory of vector-spaces such that either the dimension is at least 2, or the scalar field contains $\sqrt{2}$. As in 7, T_0 does not have the Amalgamation Property. The theory T_1 is $\forall\exists$. It also has the model $(\mathbb{Q} * \mathbf{a} \oplus \mathbb{Q} * \mathbf{b}, \mathbf{a}, \mathbf{b})$, and $\mathbb{Q} * \mathbf{a} \oplus \mathbb{Q} * \mathbf{b}$ embeds in the model $\mathbb{Q}(\sqrt{2}, \sqrt{3}) * \mathbf{a}$ of T_0 when we let $\mathbf{b} = \sqrt{3} * \mathbf{a}$; but then the latter space embeds in no space in which \mathbf{a} and \mathbf{b} are as required by $(**)$. So A fails. Finally, T_1 has a model-companion, axiomatized by VS_2^* together with

$$\exists x \exists y (x * \mathbf{a} + y * \mathbf{b} = \mathbf{0} \wedge 2x^2 = y^2 \wedge x \neq 0);$$

and T_0 has a model-companion, which is just VS_2^* ; so B holds.

9. T_0 and T_1 are both VS_1 .

10. $T_1 = DF_p$, and T_0 is the reduct to \mathcal{S}_f , namely field-theory in characteristic p .

11. T_0 and T_1 are both field-theory. \square

Now let $\omega\text{-DCF}_0 = \bigcup_{m \in \omega} m\text{-DCF}_0$. We obtain a positive application of Theorem 1.

Theorem 6. *For all m in ω ,*

$$m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0.$$

Therefore $\omega\text{-DCF}_0$ has a model-companion, which is $\omega\text{-DCF}_0$. This theory admits full elimination of quantifiers, is complete, and is properly stable.

Proof. Suppose $(L, \partial_0, \dots, \partial_{m-1})$ is a model of $m\text{-DCF}_0$, and L has a subfield K that is closed under the ∂_i (where $i < m$), and there is also a derivation ∂_m on K such that $(K, \partial_0 \upharpoonright K, \dots, \partial_{m-1} \upharpoonright K, \partial_m)$ is a model of $(m+1)\text{-DCF}_0$. We shall include $(L, \partial_0, \dots, \partial_{m-1})$ in another model of $m\text{-DCF}_0$, namely a model that expands to a model of $(m+1)\text{-DCF}_0$ that includes $(K, \partial_0, \dots, \partial_m)$. Thus condition A of Theorem 5 will hold, and therefore condition B will hold: this means $m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0$. Since m is arbitrary, it will follow by Theorem 1 that $\omega\text{-DCF}_0$ is the model-companion of $\omega\text{-DF}_0$.

If $K = L$, we are done. Suppose $a \in L \setminus K$. We shall define a differential field $(K\langle a \rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m)$, where $a \in K\langle a \rangle$, and for each i in m ,

$$\tilde{\partial}_i \upharpoonright K\langle a \rangle \cap L = \partial_i \upharpoonright K\langle a \rangle \cap L, \quad (\dagger\dagger)$$

and $\tilde{\partial}_m \upharpoonright K = \partial_m$. Then we shall be able to repeat the process, in case $L \not\subseteq K\langle a \rangle$: we can work with an element of $L \setminus K\langle a \rangle$ as we did with a . Ultimately we shall obtain the desired model of $(m+1)\text{-DF}_0$ with reduct that includes $(L, \partial_0, \dots, \partial_{m-1})$.

Considering ω^{m+1} as the set of $(m+1)$ -tuples of natural numbers, we shall have

$$K\langle a \rangle = K(a^\sigma : \sigma \in \omega^{m+1}),$$

where

$$a^\sigma = \tilde{\partial}_0^{\sigma(0)} \dots \tilde{\partial}_m^{\sigma(m)} a. \quad (\dagger\dagger)$$

In particular then, by $(\dagger\dagger)$, we must have

$$\sigma(m) = 0 \implies a^\sigma = \partial_0^{\sigma(0)} \dots \partial_{m-1}^{\sigma(m-1)} a.$$

Using this rule, we make the definition

$$K_1 = K(a^\sigma : \sigma(m) = 0).$$

We may assume that the derivations $\tilde{\partial}_i$ have been defined so far that

$$i < m \implies \tilde{\partial}_i \upharpoonright K_1 = \partial_i \upharpoonright K_1, \quad \tilde{\partial}_m \upharpoonright K = \partial_m \upharpoonright K. \quad (\S\S)$$

Then $(\dagger\dagger)$ holds when $\sigma(m) < 1$.

Now suppose that, for some positive j in ω , we have been able to define the field $K(a^\sigma : \sigma(m) < j)$, and for each i in m , we have been able to define $\tilde{\partial}_i$ as a derivation on this field, and we have been able to define $\tilde{\partial}_m$ as a derivation from $K(a^\sigma : \sigma(m) < j-1)$ to $K(a^\sigma : \sigma(m) < j)$, so that $(\S\S)$ holds, and $(\dagger\dagger)$ holds when $\sigma(m) < j$. We want to define the a^σ such that $\sigma(m) = j$, and we want to be able to extend the derivations $\tilde{\partial}_i$ appropriately.

If $i < m+1$, then, as in [11, §4.1], we let \mathbf{i} denote the characteristic function of $\{i\}$ on $m+1$: that is, \mathbf{i} will be the element of ω^{m+1} that takes the value 1 at i and 0 elsewhere. Considered as a product structure, ω^{m+1} inherits from ω the binary operations $-$ and $+$. For each i in $m+1$, we have a derivation $\tilde{\partial}_i$ from $K(a^\sigma : (\sigma + \mathbf{i})(m) < j)$ to $K(a^\sigma : \sigma(m) < j)$ such that $(\S\S)$ holds, and also, if $\sigma(m) < j$, then

$$\sigma(i) > 0 \implies \tilde{\partial}_i a^{\sigma - \mathbf{i}} = a^\sigma. \quad (\P\P)$$

We now define the a^σ , where $\sigma(m) = j$, so that, first of all, we can extend $\tilde{\partial}_m$ so that $(\P\P)$ holds when $\sigma(m) = j$ and $i = m$; but we must also ensure that $(\P\P)$ can hold also when $\sigma(m) = j$ and $i < m$. To do this, we shall have to make an inductive hypothesis, which is vacuously satisfied when $j = 1$. We shall also proceed recursively again. More precisely, we shall refine the recursion that we are already engaged in.

We well-order the elements σ of ω^{m+1} by the linear ordering \triangleleft determined by the left-lexicographic ordering of the $(m+1)$ -tuples

$$(\sigma(m), \sigma(0) + \dots + \sigma(m-1), \sigma(0), \sigma(1), \dots, \sigma(m-2)).$$

Then $(\omega^{m+1}, \triangleleft)$ has the order-type of the ordinal ω^2 . This is a difference from the linear ordering defined in [11, §4.1] and elsewhere. However, for all σ and τ in ω^{m+1} , and all i in $m+1$, we still have

$$\sigma \triangleleft \tau \implies \sigma + \mathbf{i} \triangleleft \tau + \mathbf{i}.$$

We have assumed that, when $\tau = (0, \dots, 0, j)$, we have the field $K(a^\sigma : \sigma \triangleleft \tau)$, together with, for each i in $m+1$, a derivation $\tilde{\partial}_i$ from $K(a^\xi : \xi + \mathbf{i} \triangleleft \tau)$ to $K(a^\xi : \xi \triangleleft \tau)$ such that $(\S\S)$ holds, and also, if $\sigma \triangleleft \tau$, then $(\P\P)$ holds. We have

noted that we *can* have all of this when $\tau = (0, \dots, 0, 1)$. Suppose we have all of this for *some* τ in ω^{m+1} such that $(0, \dots, 0, 1) \trianglelefteq \tau$, that is, $\tau(m) > 0$. We want to define the extension $K(a^\sigma : \sigma \trianglelefteq \tau)$ of $K(a^\sigma : \sigma \triangleleft \tau)$ so that we can extend the $\tilde{\partial}_i$ appropriately. For defining a^τ , there are two cases to consider. We use the rules for derivations gathered, for example, in [10, Fact 1.1].

1. If $a^{\tau-\mathbf{m}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \mathbf{m})$, then the derivative $\tilde{\partial}_m a^{\tau-\mathbf{m}}$ is determined as an element of $K(a^\xi : \xi \triangleleft \tau)$; we let a^τ be this element.
2. If $a^{\tau-\mathbf{m}}$ is not algebraic over $K(a^\xi : \xi \triangleleft \tau - \mathbf{m})$, then we let a^τ be transcendental over $L(a^\xi : \xi \triangleleft \tau)$. We are then free to define $\tilde{\partial}_m a^{\tau-\mathbf{m}}$ as a^τ . (We require a^τ to be transcendental over $L(a^\xi : \xi \triangleleft \tau)$, and not just over $K(a^\xi : \xi \triangleleft \tau)$, so that we can establish $(\dagger\dagger)$ later.)

We now check that, when $i < m$ and $\tau(i) > 0$, we can define $\tilde{\partial}_i a^{\tau-\mathbf{i}}$ as a^τ . Here we make the inductive hypothesis mentioned above, namely that the foregoing two-part definition of a^τ was already used to define $a^{\tau-\mathbf{i}}$. Again we consider two cases.

1. Suppose $a^{\tau-\mathbf{i}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \mathbf{i})$. Then $\tilde{\partial}_i a^{\tau-\mathbf{i}}$ is determined as an element of $K(a^\xi : \xi \triangleleft \tau)$. Thus the value of the bracket $[\tilde{\partial}_i, \tilde{\partial}_m]$ at $a^{\tau-\mathbf{i}-\mathbf{m}}$ is determined: indeed, we have

$$[\tilde{\partial}_i, \tilde{\partial}_m] a^{\tau-\mathbf{i}-\mathbf{m}} = \tilde{\partial}_i \tilde{\partial}_m a^{\tau-\mathbf{i}-\mathbf{m}} - \tilde{\partial}_m \tilde{\partial}_i a^{\tau-\mathbf{i}-\mathbf{m}} = \tilde{\partial}_i a^{\tau-\mathbf{i}} - a^\tau.$$

By inductive hypothesis, since $a^{\tau-\mathbf{i}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \mathbf{i})$, also $a^{\tau-\mathbf{i}-\mathbf{m}}$ must be algebraic over $K(a^\xi : \xi \triangleleft \tau - \mathbf{i} - \mathbf{m})$. Since the bracket is 0 on this field, it must be 0 at $a^{\tau-\mathbf{i}-\mathbf{m}}$ as well [11, Lem. 4.2].

2. If $a^{\tau-\mathbf{i}}$ is transcendental over $K(a^\xi : \xi \triangleleft \tau - \mathbf{i})$, then, since we are given $\tilde{\partial}_i$ as a derivation whose domain is this field, we are free to define $\tilde{\partial}_i a^{\tau-\mathbf{i}}$ as a^τ .

Thus we have obtained $K(a^\xi : \xi \trianglelefteq \tau)$ as desired. By induction, we obtain the differential field $(K(a^\sigma : \sigma \in \omega^{m+1}), \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ such that $(\dagger\dagger)$ and $(\S\S)$ hold.

It remains to check that $(\dagger\dagger)$ holds. It is enough to show

$$K\langle a \rangle \cap L \subseteq K_1. \quad (***)$$

(We have the reverse inclusion.) Suppose $\tau \in \omega^{m+1}$ and $\tau(m) > 0$. By the definition of a^τ ,

$$a^\tau \in K(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}} \implies a^\tau \in K(a^\sigma : \sigma \triangleleft \tau), \quad (\dagger\dagger\dagger)$$

$$a^\tau \notin K(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}} \implies a^\tau \notin L(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}}. \quad (\dagger\dagger\dagger)$$

Suppose $b \in K\langle a \rangle \cap L$. Since $b \in K\langle a \rangle$, we have, for some τ in ω^{m+1} , that b is a rational function over K_1 of those a^σ such that $\mathbf{m} \trianglelefteq \sigma \trianglelefteq \tau$. But then, by $(\dagger\dagger\dagger)$, we do not need any a^σ that is algebraic over $K(a^\xi : \xi \triangleleft \sigma)$, since it actually belongs to this field. When we throw out all such a^σ , then, by $(\dagger\dagger\dagger)$, those that remain are algebraically independent over L . Thus we have

$$b \in K_1(a^{\sigma_0}, \dots, a^{\sigma_{n-1}}) \cap L$$

for some σ_j in ω^{m+1} such that $(a^{\sigma_0}, \dots, a^{\sigma_{n-1}})$ is algebraically independent over L . Therefore we may assume $n = 0$, and $b \in K_1$. Thus $(***)$ holds, and we have the differential field $(K\langle a \rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ fully as desired.

We have to be able to repeat this construction, in case $L \not\subseteq K\langle a \rangle$. If $b \in L \setminus K\langle a \rangle$, we have to be able to construct $K\langle a, b \rangle$, and so on. Let $L\langle a \rangle$ be the compositum of $K\langle a \rangle$ and L . Since $m\text{-DF}_0$ has the Amalgamation Property, we can extend the $\tilde{\partial}_i$, where $i < m$, to commuting derivations on the field $L\langle a \rangle$ that extend the original ∂_i on L . Thus we have a model $(L\langle a \rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of $m\text{-DF}_0$ and a model $(K\langle a \rangle, \tilde{\partial}_0 \upharpoonright K\langle a \rangle, \dots, \tilde{\partial}_{m-1} \upharpoonright K\langle a \rangle, \tilde{\partial}_m)$ of $(m+1)\text{-DF}_0$ that include, respectively, the models that we started with. Now we can continue as before, ultimately extending

the domain of $\tilde{\partial}_m$ to include all of L . At limit stages of this process, we take unions, which is no problem, since $m\text{-DF}_0$ and $(m+1)\text{-DF}_0$ are $\forall\exists$.

Therefore $\omega\text{-DF}_0$ has the model-companion $\omega\text{-DCF}_0$. Since the $m\text{-DCF}_0$ have the properties of quantifier-elimination, completeness, and stability [6], the observations of Medvedev noted earlier allow us to conclude that $\omega\text{-DCF}_0$ also has these properties. Although each $m\text{-DCF}_0$ is actually ω -stable, $\omega\text{-DCF}_0$ is not even superstable, since if A is a set of constants (in the sense that all of their derivatives are 0), then as σ ranges over A^ω , the sets $\{\partial_m x = \sigma(m) : m \in \omega\}$ belong to distinct complete types. \square

In the foregoing proof, we cannot use Condition A of Theorem 5 in the stronger form in which the structure \mathfrak{C} is required to be a mere *expansion* to \mathcal{S}_1 of \mathfrak{B} :

Theorem 7. *If $m > 0$, there is a model \mathfrak{K} of $(m+1)\text{-DF}_0$ with a reduct that is included in a model \mathfrak{L} of $m\text{-DF}_0$, while \mathfrak{L} does not expand to a model of $(m+1)\text{-DF}_0$ that includes \mathfrak{K} .*

Proof. We generalize the example of [4] repeated in [9, Ex. 1.2, p. 927]. Suppose K is a pure transcendental extension $\mathbb{Q}(a^\sigma : \sigma \in \omega^{m+1})$ of \mathbb{Q} . We make this into a model of $(m+1)\text{-DF}_0$ by requiring $\partial_i a^\sigma = a^{\sigma+i}$ in each case. Let L be the pure transcendental extension $K(b^\tau : \tau \in \omega^{m-1})$ of K . We make this into a model of $m\text{-DF}_0$ by extending the ∂_i so that, if $i < m-1$, we have $\partial_i b^\tau = b^{\tau+i}$, while $\partial_{m-1} b^\tau$ is the element $a^{(\tau,0,0)}$ of K . Note that indeed if $i < m-1$, then

$$[\partial_i, \partial_{m-1}]b^\tau = \partial_i a^{(\tau,0,0)} - \partial_{m-1} b^{\tau+i} = 0.$$

Suppose, if possible, ∂_m extends to L as well so as to commute with the other ∂_i . Then for any τ in ω^{m-1} we have $\partial_m b^\tau = f(b^\xi : \xi \in \omega^{m-1})$ for some polynomial f over K . But then, writing $\partial_\eta f$ for the derivative of f with respect to the variable indexed by η , we have, as by [10, Fact 1.1(0)],

$$\begin{aligned} a^{(\tau,0,1)} &= \partial_m \partial_{m-1} b^\tau \\ &= \partial_{m-1} \partial_m b^\tau \\ &= \partial_{m-1} (f(b^\xi : \xi \in \omega^{m-1})) \\ &= \sum_{\eta \in \omega^{m-1}} \partial_\eta f(b^\xi : \xi \in \omega^{m-1}) \cdot a^{(\eta,0,0)} + f^{\partial_{m-1}}(b^\xi : \xi \in \omega^{m-1}), \end{aligned}$$

where the sum has only finitely many nonzero terms. The polynomial expression $f^{\partial_{m-1}}(b^\xi : \xi \in \omega^{m-1})$ cannot have $a^{(\tau,0,1)}$ as a constant term, since this is not $\partial_{m-1} x$ for any x in K . Thus we have obtained an algebraic relation among the b^σ and a^τ ; but there can be no such relation. \square

Finally, the union of a chain of non-companionable theories may be companionable:

Theorem 8. *In the signature $\{f\} \cup \{c_k : k \in \omega\}$, where f is a singular operation-symbol and the c_k are constant-symbols, let T_0 be axiomatized by the sentences*

$$\forall x \forall y (fx = fy \rightarrow x = y)$$

and, for each k in ω ,

$$\forall x (f^{k+1}x \neq x), \quad \forall x (fx = c_k \rightarrow x = c_{k+1}), \quad fc_{k+2} = c_{k+1} \rightarrow fc_{k+1} = c_k.$$

For each n in ω , let T_{n+1} be axiomatized by

$$T_n \cup \{fc_{n+1} = c_n\}.$$

Then

- (1) each T_n is universally axiomatized, and a fortiori $\forall\exists$, so it does have existentially closed models;
- (2) each T_n has the Amalgamation Property;
- (3) every existentially closed model of T_{n+1} is an existentially closed model of T_n ;
- (4) no T_n is companionable;
- (5) $\bigcup_{n \in \omega} T_n$ is companionable.

Proof. Let \mathfrak{A}_m be the model of T_0 with universe $\omega \times \omega$ such that

$$f^{\mathfrak{A}_m}(k, \ell) = (k, \ell + 1), \quad c_k^{\mathfrak{A}_m} = \begin{cases} (k - m, 0), & \text{if } k > m, \\ (0, m - k), & \text{if } k \leq m. \end{cases}$$

Let \mathfrak{A}_ω be the model of T_0 with universe \mathbb{Z} such that

$$f^{\mathfrak{A}_\omega} k = k + 1, \quad c_k^{\mathfrak{A}_\omega} = -k.$$

Then \mathfrak{A}_m is a model of each T_k such that $k \leq m$; and \mathfrak{A}_ω is a model of each T_k . Moreover, each model of T_k consists of a copy of some \mathfrak{A}_β such that $k \leq \beta \leq \omega$, along with some (or no) disjoint copies of ω and \mathbb{Z} in which f is interpreted as $x \mapsto x + 1$. Conversely, every structure of this form is a model of T_k . The β such that \mathfrak{A}_β embeds in a given model of T_k is uniquely determined by that model. Consequently T_k has the Amalgamation Property. Also, a model of T_k is an existentially closed model if and only if it includes no copies of ω (outside the embedded \mathfrak{A}_β): This establishes that every existentially closed model of T_{k+1} is an existentially closed model of T_k .

The existentially closed models of T_k are those models that omit the type $\{\forall y \, fy \neq x\} \cup \{x \neq c_j : j \in \omega\}$. In particular, \mathfrak{A}_m is an existentially closed model of T_k , if $k \leq m$; but \mathfrak{A}_m is elementarily equivalent to a structure that realizes the given type. Thus T_k is not companionable.

Finally, the model-companion of $\bigcup_{k \in \omega} T_k$ is axiomatized by this theory, together with $\forall x \exists y \, fy = x$. \square

REFERENCES

- [1] Zoé Chatzidakis and Ehud Hrushovski, *Model theory of difference fields*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 2997–3071. MR 2000f:03109
- [2] Paul Eklof and Gabriel Sabbagh, *Model-completions and modules*, Ann. Math. Logic **2** (1970/1971), no. 3, 251–295. MR MR0277372 (43 #3105)
- [3] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR 94e:03002
- [4] Joseph Johnson, Georg M. Reinhart, and Lee A. Rubel, *Some counterexamples to separation of variables*, J. Differential Equations **121** (1995), no. 1, 42–66. MR 96g:35006
- [5] Angus Macintyre, *Generic automorphisms of fields*, Ann. Pure Appl. Logic **88** (1997), no. 2–3, 165–180, Joint AILA-KGS Model Theory Meeting (Florence, 1995). MR 99c:03046
- [6] Tracey McGrail, *The model theory of differential fields with finitely many commuting derivations*, J. Symbolic Logic **65** (2000), no. 2, 885–913. MR 2001h:03066
- [7] Alice Medvedev, *$\mathbb{Q}ACFA$* , Talk given at Recent Developments in Model Theory, Oléron, France, <http://modeltheory2011.univ-lyon1.fr/abstracts.html>, June 2011.
- [8] ———, *$\mathbb{Q}ACFA$* , preprint, <http://math.berkeley.edu/~alice/grouplessqacfa.pdf>, November 2012.
- [9] David Pierce, *Differential forms in the model theory of differential fields*, J. Symbolic Logic **68** (2003), no. 3, 923–945. MR 2004d:03047
- [10] ———, *Geometric characterizations of existentially closed fields with operators*, Illinois J. Math. **48** (2004), no. 4, 1321–1343. MR MR2114160
- [11] ———, *Fields with several commuting derivations*, ArXiv e-prints (2007).
- [12] ———, *Model-theory of vector-spaces over unspecified fields*, Arch. Math. Logic **48** (2009), no. 5, 421–436. MR MR2505433

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